continuation of Vector Spaces 2

$$\mathbb{P}_2(\mathbb{R}) = \left\{ p(x) \in \mathbb{P}_2 \,/\, p(x) = ax^2 + bx + c \quad \forall a,b,c \in \mathbb{R} \right\}$$

$$\mathbb{R}^3(\mathbb{R}) = \left\{ \bar{p} \in \mathbb{R}^3 \,/\, \bar{p} = (a, b, c) \quad \forall a,b,c \in \mathbb{R} \right\}$$

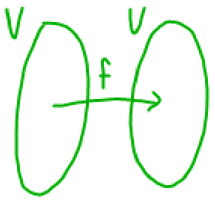
$$B = \{ \underset{(1,0,0)}{x^2}, \underset{(0,1,0)}{x}, \underset{(0,0,1)}{1} \}$$

$$B' = \{ \underset{(1,1,1)}{x^2+x+1}, \underset{(1,1,0)}{x^2+x}, \underset{(1,0,0)}{x^2} \}$$

$$F_B = \begin{pmatrix} | & | & | \\ | & | & | \\ | & | & | \end{pmatrix}$$

$$\boxed{F_{B'} \ ?}$$

Change of base for an endomorphism matrix



$$+ \; B \underset{C^{-1}}{\overset{C}{\rightleftarrows}} B' \Rightarrow F_{B'} = C^{-1} F_B C$$

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

x^2+x+1 $\quad x^2+x$ $\quad x^2$

expressed in B

$$C^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$F_{B'} = C^{-1} F_B C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ \dots \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ \dots \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 0 & 0 \\ \dots \end{pmatrix}$$

Working in $\mathbb{R}^3(\mathbb{R})$

We have an ENDOMORPHISM $f(x^1,x^2,x^3) = (x^1+x^2+x^3,\ x^1+x^2+x^3,\ x^1+x^2+x^3)$

We have a base $B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ and another base $B' = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ where $\begin{cases} \bar{u}_1 = \bar{e}_1 + \bar{e}_2 + \bar{e}_3 \\ \bar{u}_2 = \bar{e}_1 + \bar{e}_2 \\ \bar{u}_3 = \bar{e}_1 \end{cases}$

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

Matrix operation works in the opposite direction to this

$$B \underset{C^{-1}}{\overset{C}{\rightleftarrows}} B'$$

$$F_B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$C = \begin{array}{c} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ \bar{u}_1 \ \ \bar{u}_2 \ \ \bar{u}_3 \end{array}$$

$\bar{u}_1\ \bar{u}_2\ \bar{u}_3$ expressed in B

$$\bar{v} = (1,2,3)_B = \bar{e}_1 + 2\bar{e}_2 + 3\bar{e}_3$$

$$F_{B'} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$C^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}}_{C^{-1}} \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_B}_{\bar{v}} = \underbrace{\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}_{B'}}_{\bar{v}}$$

$$\bar{v} = (3,-1,-1)_{B'} = 3\bar{u}_1 - \bar{u}_2 - \bar{u}_3$$

$f(\bar{v}) = \bar{w}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_B = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}_B$$

Vector Spaces 3

Given a homomorfism from $\mathbb{R}^3$ to $\mathbb{R}^2$ that works $f(x^1,x^2,x^3)=(x^1+x^3,\ x^2+x^3)$, calculate:

1. $\ker(f)$ and $\operatorname{Im}(f)$ (giving base and dimension).
2. Calculate two bases from $\mathbb{R}^3$ and $\mathbb{R}^2$ so that f in those bases has the expression $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$B_1=\{\bar{e}_1,\bar{e}_2,\bar{e}_3\}\subset\mathbb{R}^3$

$B_2=\{\bar{u}_1,\bar{u}_2\}\subset\mathbb{R}^2$

Bases of reference for the homomorphism

$f:\mathbb{R}^3\longrightarrow\mathbb{R}^2$

$$\boxed{\begin{array}{ll} \bar{e}_1=(1,0,0)_{B_1} & \bar{u}_1=(1,0)_{B_2} \\ \bar{e}_2=(0,1,0)_{B_1} & \bar{u}_2=(0,1)_{B_2} \\ \bar{e}_3=(0,0,1)_{B_1} & \end{array}}$$

$f(\bar{e}_1)=(1+0,\ 0+0)=(1,0)_{B_2}$

$f(\bar{e}_2)=(0+0,\ 1+0)=(0,1)_{B_2}$

$f(\bar{e}_3)=(0+1,\ 0+1)=(1,1)_{B_2}$

$$F_{B_1B_2}=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$f(\bar{e}_1)$ $f(\bar{e}_2)$ $f(\bar{e}_3)$ expressed in B_2

$\dim(\operatorname{Im}(f))=\operatorname{Rg}(F_{B_1B_2})=2=\dim(\mathbb{R}^2)\longrightarrow \operatorname{Im}(f)=\mathbb{R}^2\longrightarrow B_{\operatorname{Im}}=B_2=\{\bar{u}_1,\bar{u}_2\}$

$\begin{cases} x^1+x^3=0 \\ \ldots \\ x^3=\gamma \end{cases}$ $B_{\ker}$

2. $F_{B_1B_2} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$f(\bar{e}_1)$ $f(\bar{e}_2)$ $f(\bar{e}_3)$ expressed in B_2

$F_{B_3B_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$f(\bar{a}_1)$ $f(\bar{a}_2)$ $f(\bar{a}_3)$ expressed in B_4

$f(\bar{a}_1) = \bar{b}_1$
$f(\bar{a}_2) = \bar{b}_2$
$f(\bar{a}_3) = \bar{0} \longrightarrow \bar{a}_3 \in Ker(f)$

$\boxed{\begin{array}{ll} \bar{e}_1 = (1,0,0)_{B_1} & \bar{u}_1 = (1,0)_{B_2} \\ \bar{e}_2 = (0,1,0)_{B_1} & \bar{u}_2 = (0,1)_{B_2} \\ \bar{e}_3 = (0,0,1)_{B_1} & \end{array}}$

$B_3 = \{\bar{a}_1, \bar{a}_2, \bar{a}_3\}$?

$B_4 = \{\bar{b}_1, \bar{b}_2\}$?

Conditions that B_3 and B_4 must have:

1. $\bar{a}_1, \bar{a}_2$ and $\bar{a}_3$ must be L.I.
2. $\bar{b}_1$ and $\bar{b}_2$ must be L.I.
3. $\bar{a}_3 \in Ker(f)$ ⟶ MOST RESTRICTIVE
4. $\bar{a}_1$ and $\bar{a}_2$ transform into L.I. vectors.

$B_{Ker} = \{(-1,-1,1)\}$

We start off with ③ and establish:

$\bar{a}_3 = (-1,-1,1)_{B_1}$ for example

So now we pick an $\bar{a}_1$ and $\bar{a}_2$ L.I. with $\bar{a}_3$ to cover ①:

$\bar{a}_1 = (1,0,0)_{B_1}$
$\bar{a}_2 = (0,1,0)_{B_1}$

$\begin{vmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1 \neq 0 \rightarrow$ they are L.I.

$\boxed{\begin{array}{l} \bar{a}_1 = (1,0,0)_{B_1} \\ \bar{a}_2 = (0,1,0)_{B_1} \\ \bar{a}_3 = (-1,-1,1)_{B_1} \\ \bar{b}_1 = (1,0)_{B_2} \\ \bar{b}_2 = (\ldots) \end{array}}$

We ... from $\bar{a}_1$ and $\bar{a}_2$ to see if the become L.I. vector and cover ② and ④:

$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{B_2} \longrightarrow \bar{b}_1 = \ldots$